

AP[®] Calculus BC 2002 Scoring Guidelines

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Question 1

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

- (a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.
- (b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.
- (c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \leq x \leq 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \leq x \leq 1$. Show the analysis that leads to your answers.

(a) Area = $\int_{\frac{1}{2}}^1 (e^x - \ln x) dx = 1.222$ or 1.223

2 { 1 : integral
1 : answer

(b) Volume = $\pi \int_{\frac{1}{2}}^1 ((4 - \ln x)^2 - (4 - e^x)^2) dx$
= 7.515π or 23.609

4 { 1 : limits and constant
2 : integrand
< -1 > each error
Note: 0/2 if not of the form
 $k \int_a^b (R(x)^2 - r(x)^2) dx$
1 : answer

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$
 $x = 0.567143$

3 { 1 : considers $h'(x) = 0$
1 : identifies critical point
and endpoints as candidates
1 : answers

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$

$h(0.5) = 2.3418$

$h(1) = 2.718$

The absolute minimum is 2.330.

The absolute maximum is 2.718.

Note: Errors in computation come off the third point.

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Question 2

The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{(t^2 - 24t + 160)}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{(t^2 - 38t + 370)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \leq t \leq 23$, the hours during which the park is open. At time $t = 9$, there are no people in the park.

- (a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round answer to the nearest whole number.
- (b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.
- (c) Let $H(t) = \int_9^t (E(x) - L(x)) dx$ for $9 \leq t \leq 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.
- (d) At what time t , for $9 \leq t \leq 23$, does the model predict that the number of people in the park is a maximum?

(a) $\int_9^{17} E(t) dt = 6004.270$
6004 people entered the park by 5 pm.

(b) $15 \int_9^{17} E(t) dt + 11 \int_{17}^{23} E(t) dt = 104048.165$
The amount collected was \$104,048.

or
 $\int_{17}^{23} E(t) dt = 1271.283$
1271 people entered the park between 5 pm and 11 pm, so the amount collected was
 $\$15 \cdot (6004) + \$11 \cdot (1271) = \$104,041.$

(c) $H'(17) = E(17) - L(17) = -380.281$
There were 3725 people in the park at $t = 17$.
The number of people in the park was decreasing at the rate of approximately 380 people/hr at time $t = 17$.

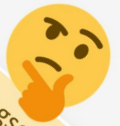
(d) $H'(t) = E(t) - L(t) = 0$
 $t = 15.794$ or 15.795

$\left\{ \begin{array}{l} 1 : \text{limits} \\ 3 \left\{ \begin{array}{l} 1 : \text{integrand} \\ 1 : \text{answer} \end{array} \right. \end{array} \right.$

1 : setup

$\left\{ \begin{array}{l} 1 : \text{value of } H'(17) \\ 2 : \text{meanings} \\ 3 \left\{ \begin{array}{l} 1 : \text{meaning of } H(17) \\ 1 : \text{meaning of } H'(17) \\ < -1 > \text{ if no reference to } t = 17 \end{array} \right. \end{array} \right.$

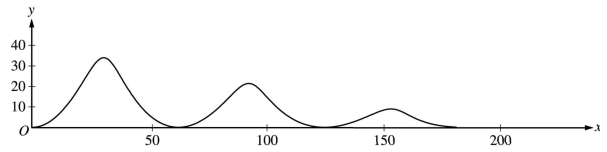
$\left\{ \begin{array}{l} 1 : E(t) - L(t) = 0 \\ 2 \left\{ \begin{array}{l} 1 : \text{answer} \end{array} \right. \end{array} \right.$



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Question 3

The figure above shows the path traveled by a roller coaster car over the time interval $0 \leq t \leq 18$ seconds. The position of the car at time t seconds can be modeled parametrically by $x(t) = 10t + 4 \sin t$, $y(t) = (20 - t)(1 - \cos t)$,



where x and y are measured in meters. The derivatives of these functions are given by

$$x'(t) = 10 + 4 \cos t, \quad y'(t) = (20 - t)\sin t + \cos t - 1.$$

- Find the slope of the path at time $t = 2$. Show the computations that lead to your answer.
- Find the acceleration vector of the car at the time when the car's horizontal position is $x = 140$.
- Find the time t at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.
- For $0 < t < 18$, there are two times at which the car is at ground level ($y = 0$). Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

(a) Slope = $\left. \frac{dy}{dx} \right|_{t=2} = \frac{y'(2)}{x'(2)} = \frac{18 \sin 2 + \cos 2 - 1}{10 + 4 \cos 2}$
 $= 1.793$ or 1.794

(b) $x(t) = 10t + 4 \sin t = 140$; $t_0 = 13.647083$
 $x''(t_0) = -3.529$, $y''(t_0) = 1.225$ or 1.226
 Acceleration vector is $\langle -3.529, 1.225 \rangle$
 or $\langle -3.529, 1.226 \rangle$

(c) $y'(t) = (20 - t)\sin t + \cos t - 1 = 0$
 $t_1 = 3.023$ or 3.024 at maximum height
 Speed = $\sqrt{(x'(t_1))^2 + (y'(t_1))^2} = |x'(t_1)|$
 $= 6.027$ or 6.028

(d) $y(t) = 0$ when $t = 2\pi$ and $t = 4\pi$
 Average speed = $\frac{1}{2\pi} \int_{2\pi}^{4\pi} \sqrt{(x'(t))^2 + (y'(t))^2} dt$
 $= \frac{1}{2\pi} \int_{2\pi}^{4\pi} \sqrt{(10 + 4 \cos t)^2 + ((20 - t)\sin t + \cos t - 1)^2} dt$

1 : answer using $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

2 { 1 : identifies acceleration vector as derivative of velocity vector
 1 : computes acceleration vector when $x = 140$

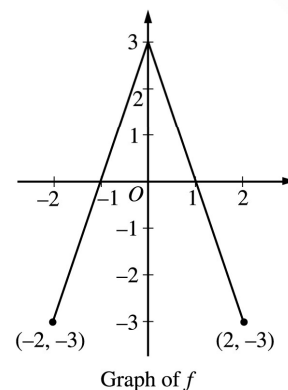
3 { 1 : sets $y'(t) = 0$
 1 : selects first $t > 0$
 1 : speed

3 { 1 : $t = 2\pi, t = 4\pi$
 1 : limits and constant
 1 : integrand

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Question 4

The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t) dt$.



- Find $g(-1)$, $g'(-1)$, and $g''(-1)$.
- For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.
- For what values of x in the open interval $(-2, 2)$ is the graph of g concave down? Explain your reasoning.
- On the axes provided, sketch the graph of g on the closed interval $[-2, 2]$.

(a) $g(-1) = \int_0^{-1} f(t) dt = -\int_{-1}^0 f(t) dt = -\frac{3}{2}$
 $g'(-1) = f(-1) = 0$
 $g''(-1) = f'(-1) = 3$

$$3 \left\{ \begin{array}{l} 1 : g(-1) \\ 1 : g'(-1) \\ 1 : g''(-1) \end{array} \right.$$

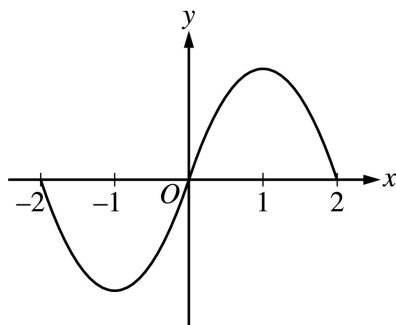
(b) g is increasing on $-1 < x < 1$ because $g'(x) = f(x) > 0$ on this interval.

$$2 \left\{ \begin{array}{l} 1 : \text{interval} \\ 1 : \text{reason} \end{array} \right.$$

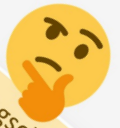
(c) The graph of g is concave down on $0 < x < 2$ because $g''(x) = f'(x) < 0$ on this interval.
 or
 because $g'(x) = f(x)$ is decreasing on this interval.

$$2 \left\{ \begin{array}{l} 1 : \text{interval} \\ 1 : \text{reason} \end{array} \right.$$

(d)



$$2 \left\{ \begin{array}{l} 1 : g(-2) = g(0) = g(2) = 0 \\ 1 : \text{appropriate increasing/decreasing} \\ \text{and concavity behavior} \\ < -1 > \text{vertical asymptote} \end{array} \right.$$



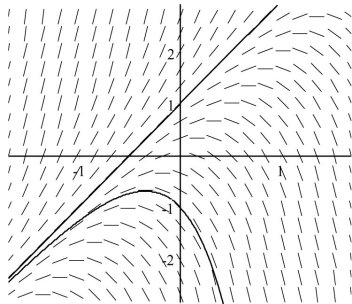
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Question 5

Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

- (a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,1)$ and sketch the solution curve that passes through the point $(0,-1)$.
- (b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1 , to approximate $f(0.2)$. Show the work that leads to your answer.
- (c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.
- (d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

(a)



- 2 { 1 : solution curve through $(0,1)$
1 : solution curve through $(0,-1)$
- Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$
 $= 1 + (2 - 0)(0.1) = 1.2$
 $f(0.2) \approx f(0.1) + f'(0.1)(0.1)$
 $\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

- 2 { 1 : Euler's method equations or equivalent table applied to (at least) two iterations
1 : Euler approximation to $f(0.2)$ (not eligible without first point)

(c) Substitute $y = 2x + b$ in the DE:
 $2 = 2(2x + b) - 4x = 2b$, so $b = 1$
 OR
 Guess $b = 1$, $y = 2x + 1$
 Verify: $2y - 4x = (4x + 2) - 4x = 2 = \frac{dy}{dx}$.

- 2 { 1 : uses $\frac{d}{dx}(2x + b) = 2$ in DE
1 : $b = 1$

(d) g has local maximum at $(0,0)$.
 $g'(0) = \left. \frac{dy}{dx} \right|_{(0,0)} = 2(0) - 4(0) = 0$, and
 $g''(x) = \frac{d^2y}{dx^2} = 2 \frac{dy}{dx} - 4$, so
 $g''(0) = 2g'(0) - 4 = -4 < 0$.

- 3 { 1 : $g'(0) = 0$
1 : shows $g''(0) = -4$
1 : conclusion

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Question 6

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \dots + \frac{(2x)^{n+1}}{n+1} + \dots$$

on its interval of convergence.

- (a) Find the interval of convergence of the Maclaurin series for f . Justify your answer.
 (b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.
 (c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

(a)
$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(2x)^{n+2}}{n+2}}{\frac{(2x)^{n+1}}{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)}{(n+2)} 2x \right| = |2x|$$

$|2x| < 1$ for $-\frac{1}{2} < x < \frac{1}{2}$

At $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges since this is the harmonic series.

At $x = -\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is $-\frac{1}{2} \leq x < \frac{1}{2}$.

(b) $f'(x) = 2 + 4x + 8x^2 + 16x^3 + \dots + 2(2x)^n + \dots$

(c) The series in (b) is a geometric series.

$$\begin{aligned} f'\left(-\frac{1}{3}\right) &= 2 + 4\left(-\frac{1}{3}\right) + 8\left(-\frac{1}{3}\right)^2 + \dots + 2\left(2 \cdot \left(-\frac{1}{3}\right)\right)^n + \dots \\ &= 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \dots + 2\left(-\frac{2}{3}\right)^n + \dots \\ &= \frac{2}{1 + \frac{2}{3}} = \frac{6}{5} \end{aligned}$$

OR

$f'(x) = \frac{2}{1-2x}$ for $-\frac{1}{2} < x < \frac{1}{2}$. Therefore,

$$f'\left(-\frac{1}{3}\right) = \frac{2}{1 + \frac{2}{3}} = \frac{6}{5}$$

- 5 {
- 1 : sets up ratio
 - 1 : computes limit of ratio
 - 1 : identifies interior of interval of convergence
 - 2 : analysis/conclusion at endpoints
 - 1 : right endpoint
 - 1 : left endpoint
 - < -1 > if endpoints not $x = \pm \frac{1}{2}$
 - < -1 > if multiple intervals

- 2 {
- 1 : first 4 terms
 - 1 : general term

- 2 {
- 1 : substitutes $x = -\frac{1}{3}$ into infinite series from (b) or expresses series from (b) in closed form
 - 1 : answer for student's series